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Software Support for Calculations in Allen's Interval Algebra

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Allen's interval algebra formally expresses temporal relations between intervals, operations on them, and reasoning about them. Many of its most interesting operations are tedious or difficult to perform by hand. This report gives a compact introduction to interval algebra and describes a software tool, `allen`*, for working with interval algebra relations. In connection with this tool, we propose a convenient notation for Allen's basic relations in which each relation is represented by a single lower or uppercase letter.*

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Allen's interval algebra formally expresses temporal relations between intervals, operations on them, and reasoning about them. Many of its most interesting operations are tedious or difficult to perform by hand. This report gives a compact introduction to interval algebra and describes a software tool, `allen`*, for working with interval algebra relations. In connection with this tool, we propose a convenient notation for Allen's basic relations in which each relation is represented by a single lower or uppercase letter.*

1 Introduction

In 1983 James F. Allen published a paper in which he proposed thirteen basic relations between time intervals; these relations are distinct, exhaustive, and qualitative [All83]:

- *distinct* because no pair of definite intervals can be related by more than one of the relations;
- *exhaustive* because every pair of definite intervals is described by one of the relations; and
- *qualitative* (rather than quantitative) because no numeric time spans are considered in describing the relations.

The basic interval relations are shown schematically in Figure 1. These relations and the operations on them form *Allen's interval algebra*.

Some of the operations on interval algebra relations and on sets of relations are difficult or tedious to do by hand. We have produced a Java command-line tool, `allen`, to perform many of these operations. The tool is publicly available at `http://www.ics.uci.edu/~alspaugh/bin/allen.jar`.

precedes	meets	overlaps	is finished by	contains	starts	equals	is started by	during	finishes	is overlapped by	is met by	is preceded by
p	m	o	F	D	s	e	S	d	f	O	M	P

Figure 1. Allen's thirteen basic relations

Relation			Converse
precedes	(p)	(P)	is preceded by
meets	(m)	(M)	is met by
overlaps	(o)	(O)	is overlapped by
is finished by	(F)	(f)	finishes
contains	(D)	(d)	during
starts	(s)	(S)	is started by
equals (e)			

Table 1. Converses of Allen's basic temporal relations

2 Allen's Interval Algebra

2.1 Basic Interval Relations

Allen's thirteen basic interval relations are illustrated in Figure 1. Each is defined graphically by a diagram relating two definite intervals a and b, with time running from left to right. For example, the first diagram shows that "a precedes b" means that interval a ends before interval b begins, with a gap separating them, and that this relation between a and b is written $a(\mathsf{p})b$; the second shows that "a meets b" means that b begins when a ends, written $a(\mathsf{m})b$;.

The basic relations are listed in Figure 1 sorted by the extent to which a begins before b, and within that by the extent to which a ends before b. We will always list them in this order (pmoFDseSdfOMP), as it makes the relations easier to remember and simplifies comparison of relations.

Six pairs of the basic relations are converses. For example, the converse of "a precedes b" is "b is preceded by a", written $a(\mathsf{p})b$ and $b(\mathsf{P})a$ respectively; whenever the first relation is true, its converse is true also. Table 1 lists the relations with each one beside its converse. The thirteenth basic relation, "equals", is its own converse. Each pair of symbols consists of the lowercase and uppercase of the same letter, e.g. (p) and (P), (m) and (M), ... The uppercase letters represent the relations Allen expressed with two letters in his notation, as 'pi', 'mi', etc. (he termed them 'inverses').

2.2 Interval Relations

The thirteen basic relations describe relations between definite intervals. Indefinite intervals whose exact relation may be uncertain are described by a list of all the basic relations that may apply. We call such a list of basic relations a *general relation*, or just a *relation*.

Example "John was not in the room when I touched the switch to turn on the light" [All83, page 837].

Let a be the interval during which John was in the room, b be the interval during which the speaker touched the light switch, and c be the interval during which the light was on. Then we can characterize the temporal relations described in the example as $a(\mathsf{pmMP})b$, that is, a precedes, meets, is met by, or is preceded by b; and $b(\mathsf{mo})c$, that is, b meets or overlaps c (see Figure 2).

There is a general relation for every combination of the thirteen basic relations: 2^{13} or 8192 of them. Each of the basic relations is a relation, of course, as are all their combinations. The full relation (pmoFDseSdfOMP) holds between two intervals about which nothing is known. The empty relation () cannot hold between two intervals, but can be the result of formal operations on interval relations. An empty relation inferred between two intervals indicates an inconsistent (unsatisfiable) collection of relations between intervals (discussed below in Section 2.4).

2.3 Operations on Interval Relations

There are five commonly encountered operations on interval relations.

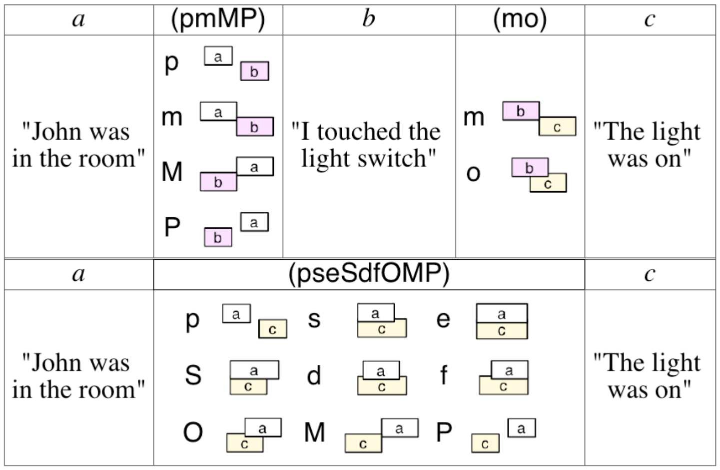

Figure 2. 'John was not in the room when I touched the switch to turn on the light"

union The *union* of two interval relations q and r, denoted $q + r$, is the set-theoretic union of the basic relations in q and r. General relations may be considered to be created through the union of their component basic relations. EXAMPLE: (pmo) + (oFD) = (pmoFD).

intersection The *intersection* of two interval relations q and r, denoted $q \wedge r$, is the set-theoretic intersection of the basic relations in q and r. EXAMPLE: (pmo) $\wedge$ (oFD) = (o).

composition Let q and r be relations on intervals a, b, and c, such that $a(q)b$ and $b(r)c$. The *composition* of q and r, denoted $q.r$, is the relation that holds between a and c, $a(q.r)c$. Composition is not commutative but is associative, and it distributes over union and intersection. Composition is the most interesting of the operations and is discussed at greater length in Section 2.4.

converse The *converse* of an interval relation is the union of the converses of its basic relations. In our notation, the converse of a basic relation r other than (e) is denoted by the same letter in the other case: (p) and (P), (m) and (M), etc. (e) ("equals") is its own converse. The converse operation is its own inverse: $!!r = r$ for any relation r. EXAMPLE: !(pmo) = (OMP).

complement The *complement* of an interval relation a, denoted $\tilde{}a$, is the set-theoretic complement of the basic relations in a. The complement operation is its own inverse: $\tilde{}\tilde{}r = r$ for any relation r. EXAMPLE: ˜(pmo) = (DFseSfdOMP).

2.4 Composition, inference, and satisfiability

There are several methods of determining the composition of two relations q and r. The first is to reason from the definitions of the basic relations involved; this is the most basic way but also the most difficult.

The second is to use a table of compositions of basic relations such as Table 2, and the fact that composition distributes over union. First look up the pairwise compositions of each basic relation in q with each basic relation in r. Then calculate the union of all those compositions; the result is the composition $q.r$. This is simpler and more mechanical than the first method, but still quite tedious and time-consuming; (pmoFD).(dfOMP) requires looking up 25 compositions in the table and then taking their union.

The third and most satisfactory way is to use a software tool such as `allen` to calculate the composition. This method is much quicker and less error-prone than the others, and was a primary motivation for the initial development of the tool.

The table of basic compositions displays many intriguing features, a few of which we will discuss briefly. There is a striking but imperfect symmetry in the table, highlighted here by giving each relation in the table a distinct color. Although there are 8192 interval relations, only 27 of them appear among the 169 entries of the table, with

.	p	m	o	F	D	s	e	S	d	f	O	M	P
p	(p)	(p)	(p)	(p)	(p)	(p)	(p)	(p)	(pmosd)	(pmosd)	(pmosd)	(pmosd)	*full*
m	(p)	(p)	(p)	(p)	(p)	(m)	(m)	(m)	(osd)	(osd)	(osd)	(Fef)	(DSOMP)
o	(p)	(p)	(pmo)	(pmo)	(pmoFD)	(o)	(o)	(oFD)	(osd)	(osd)	*concur*	(DSO)	(DSOMP)
F	(p)	(m)	(o)	(F)	(D)	(o)	(F)	(D)	(osd)	(Fef)	(DSO)	(DSO)	(DSOMP)
D	(pmoFD)	(oFD)	(oFD)	(D)	(D)	(oFD)	(D)	(D)	*concur*	(DSO)	(DSO)	(DSO)	(DSOMP)
s	(p)	(p)	(pmo)	(pmo)	(pmoFD)	(s)	(s)	(seS)	(d)	(d)	(dfO)	(M)	(P)
e	(p)	(m)	(o)	(F)	(D)	(s)	(e)	(S)	(d)	(f)	(O)	(M)	(P)
S	(pmoFD)	(oFD)	(oFD)	(D)	(D)	(seS)	(S)	(S)	(dfO)	(O)	(O)	(M)	(P)
d	(p)	(p)	(pmosd)	(pmosd)	*full*	(d)	(d)	(dfOMP)	(d)	(d)	(dfOMP)	(P)	(P)
f	(p)	(m)	(osd)	(Fef)	(DSOMP)	(d)	(f)	(OMP)	(d)	(f)	(OMP)	(P)	(P)
O	(pmoFD)	(oFD)	*concur*	(DSO)	(DSOMP)	(dfO)	(O)	(OMP)	(dfO)	(O)	(OMP)	(P)	(P)
M	(pmoFD)	(seS)	(dfO)	(M)	(P)	(dfO)	(M)	(P)	(dfO)	(M)	(P)	(P)	(P)
P	*full*	(dfOMP)	(dfOMP)	(P)	(P)	(dfOMP)	(P)	(P)	(dfOMP)	(P)	(P)	(P)	(P)

Table 2. Composition of basic interval relations

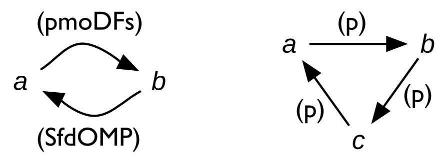

Figure 3. Two unsatisfiable collections of relations

(p) and (P) being by far the most common at 22 appearances each. Only relations containing 1, 3, 5, 9, or 13 basic relations are found. If a relation appears in the table, then its converse appears also, and (except for (s), 4 times, and (S), 3 times) each relation and its converse appear the same number of times. (e), which is of course its own converse, appears once (as the result of (e).(e)).

Composition is the mechanism by which inference among relations is done, as illustrated by the earlier example "John was not in the room when I touched the switch to turn on the light" (Figure 2). The relationship between event a "John was in the room" and event c "The light was on" is inferred from $a(\mathsf{pmMP})b$ and $b(\mathsf{mo})c$ as the composition $(\mathsf{pmMP}).(\mathsf{mo}) = (\mathsf{pseSdfOMP})$. The result (shown in the figure) can be verified by comparing each listed basic relation against the original statement.

Given a collection of relations among indefinite intervals, it is possible to mechanically infer the relation among any two intervals by following each path connecting the two intervals, calculating the composition of each path and taking the intersections of all the compositions. Not surprisingly, the number of paths and thus the complexity of the inference rises exponentially as relations are added to the collection, and in general the problem of determining the relation among two intervals is NP-complete.

It is possible for a collection of relations among intervals to be *unsatisfiable*, that is, for inference to show that a pair of intervals in the collection have the empty relation, or that an interval has a relation other than (e) with itself, indicating that there is no possible assignment of times on the real line to the intervals such that all the relations are satisfied. A simple example is the collection

$$\{a(\mathsf{pmoDFs})b, a(\mathsf{SfdOMP})b\}$$

which has two paths from a to b whose intersection $(\mathsf{pmoDFs}).(\mathsf{SfdOMP}) = ()$ indicates that a and b have no relation, signifying that the collection of relations is not satisfiable (Figure 3). A second simple example is the circuit

$$\{a(\mathsf{p})b, b(\mathsf{p})c, c(\mathsf{p})a\}$$

from which can be inferred that the relation of a to itself is $(\mathsf{p}).(\mathsf{p}).(\mathsf{p}) = (\mathsf{p})$ (a precedes itself), indicating the collection is not satisfiable.

2.5 Tractability

Satisfiability for Allen's interval algebra was shown to be intractable by Vilain *et al.* [VKB89]. Nebel and Bürckert first demonstrated the existence of a maximal tractable subalgebra of interval algebra, the "ORD-Horn" subalgebra [NB95]. A *maximal tractable subalgebra* is a subset of the full interval algebra for which:

- satisfiability is *tractable*,
- addition of any other relation would result in an intractable set (thus *maximal*), and
- the set is closed under the operations of converse, composition, intersection, and union (thus a *subalgebra*).

Krokhin *et al.* showed that there are 18 maximal tractable subalgebras, and summarized rules defining which relations are in each subalgebra [KJJ03]. For example, a relation r is an element of the $\mathcal{H}$ or ORD-Horn subalgebra if it satisfies the following rules (translated to our notation):

1^+.	If	$r \cap (\mathsf{os}) \neq ()$	and	$r \cap (\mathsf{Of}) \neq ()$,	then	$(\mathsf{d}) \subseteq r$
1^-.	If	$r \cap (\mathsf{SO}) \neq ()$	and	$r \cap (\mathsf{oF}) \neq ()$,	then	$(\mathsf{D}) \subseteq r$
2^+.	If	$r \cap (\mathsf{sd}) \neq ()$	and	$r \cap (\mathsf{FD}) \neq ()$,	then	$(\mathsf{o}) \subseteq r$
2^-.	If	$r \cap (\mathsf{SD}) \neq ()$	and	$r \cap (\mathsf{fd}) \neq ()$,	then	$(\mathsf{O}) \subseteq r$
3^+.	If	$r \cap (\mathsf{pm}) \neq ()$	and	$r \nsubseteq (\mathsf{pm})$	then	$(\mathsf{o}) \subseteq r$
3^-.	If	$r \cap (\mathsf{MP}) \neq ()$	and	$r \nsubseteq (\mathsf{MP})$	then	$(\mathsf{O}) \subseteq r$

The rules for the $\mathcal{H}$ subalgebra are the most complex of those for any of the 18 sets (most of the others have a single pair of rules).

It is clear that examining a substantial number of relations for membership in this set of 868 relations is tedious and error-prone. `allen` examines sets of relations and determines whether they are subclasses of any of the 18 subalgebras and presents counterexamples for each that is not.

3 The `allen` tool

The `allen` tool performs operations on individual relations and on collections of relations.

3.1 Operations on individual relations

The operations on individual relations include complement, composition, converse, intersection, union, and any parenthesized expression involving them. The basic relations listed in each general relation are sorted into our standard order (pmoFDseSdfOMP).

3.2 Operations on collections of relations

The operations on collections of relations include closure under complement, composition, converse, intersection, and union; determining whether a collection is already closed under any of the operations, and producing counterexamples if it is not; comparing a collection to any of the 18 maximal tractable subalgebras and identifying any relations that are not in that subalgebra; and enumerating the relations of any of the 18 maximal tractable subalgebras or of the complete interval algebra.

3.3 Other features

By default, `allen` uses the symbology presented in this report. However, it may be customized to use any single characters in place of the basic relation characters or the relation operation characters.

4 Related Work

Gennari mentions several utilities in her excellent survey, for example `aclose` [Gen98, page 202], but these utilities appear to be no longer available.

5 Conclusions and Future Work

The `allen` tool supports research into Allen's interval algebra and its applications by making it easy to calculate compositions of relations, tractability of collections of relations, and other tedious calculations.

Future work includes extending the tool to implement the path consistency algorithm and other algorithms related to satisfiability and inference, and to incorporate the engine into software tools that apply interval algebra.

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